Final Round of the 37th Mongolian Mathematical Olympiad [2004 : 413].

2. Prove that, if ABC is an acute-angled triangle, then

$$\frac{a^2+b^2}{a+b}\cdot\frac{b^2+c^2}{b+c}\cdot\frac{c^2+a^2}{c+a} \;\geq\; 16\cdot R^2\cdot r\cdot\frac{m_a}{a}\cdot\frac{m_b}{b}\cdot\frac{m_c}{c}\,.$$

Solution by Arkady Alt, San Jose, CA, USA, modified by the editor.

Using the formulas $4m_a^2 = 2(b^2+c^2)-a^2$ and

$$w_a^2 \;=\; \frac{bc(b+c+a)(b+c-a)}{(b+c)^2} \;=\; \frac{4bcs(s-a)}{(b+c)^2}\,,$$

it can be shown that

$$m_a \;\leq\; w_a\cdot\frac{b^2+c^2}{2bc}\,. \qquad (1)$$

[*Ed.:* We omit Alt's proof of (1) because the same argument was given recently by Alt in his (featured) solution to problem 2963 [2005 : 350–351], to which we refer the reader.]

Substituting $w_a = \dfrac{2\sqrt{bcs(s-a)}}{b+c}$ in (1), we get

$$m_a \;\leq\; \frac{b^2+c^2}{b+c}\sqrt{\frac{s(s-a)}{bc}}\,.$$

Letting K denote the area of $\triangle ABC$, we have $K = \sqrt{s(s-a)(s-b)(s-c)}$ (Heron's formula). We will also use the formulas $abc = 4RK = 4Rrs$. We have

$$\begin{aligned}
16R^2r\prod_{\text{cyclic}}\frac{m_a}{a} &\leq 16R^2r\prod_{\text{cyclic}}\frac{b^2+c^2}{a(b+c)}\sqrt{\frac{s(s-a)}{bc}}\\
&= \frac{16R^2rs}{(abc)^2}\sqrt{s(s-a)(s-b)(s-c)}\prod_{\text{cyclic}}\frac{b^2+c^2}{b+c}\\
&= \frac{16R^2rs}{(4RK)(4Rrs)}(K)\prod_{\text{cyclic}}\frac{b^2+c^2}{b+c} \;=\; \prod_{\text{cyclic}}\frac{b^2+c^2}{b+c}\,,
\end{aligned}$$

as required.